Noname manuscript No.
(will be inserted by the editor)

Supersolvability and freeness for ψ-graphical arrangements

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Received: date / Accepted: date

Abstract Let G be a simple graph on the vertex set $\{v_1, \ldots, v_n\}$ with edge set E. Let K be a field. The graphical arrangement $\mathcal{A}_G$ in K^n is the arrangement $x_i - x_j = 0, v_i v_j \in E$. An arrangement $\mathcal{A}$ is supersolvable if the intersection lattice $L(c(\mathcal{A}))$ of the cone $c(\mathcal{A})$ contains a maximal chain of modular elements. The second author has shown that a graphical arrangement $\mathcal{A}_G$ is supersolvable if and only if G is a chordal graph. He later considered a generalization of graphical arrangements which are called ψ-graphical arrangements. He conjectured a characterization of the supersolvability and freeness (in the sense of Terao) of a ψ-graphical arrangement. We provide a proof of the first conjecture and state some conditions on free ψ-graphical arrangements.

Keywords graphical arrangement · supersolvable arrangement · free arrangement · chordal graph

Mathematics Subject Classification (2010) 52C35 · 05C15

The first author would like to thank the China Scholarship Council for financial support. Her work was done during her visit to the Department of Mathematics, Massachusetts Institute of Technology. The second author was partially supported by the National Science Foundation under Grant No. DMS-1068625.

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1 Introduction

A finite hyperplane arrangement $\mathcal{A}$ is a finite set of affine hyperplanes in some vector space $V \cong K^n$, where K is a field. The *intersection poset* $L(\mathcal{A})$ of $\mathcal{A}$ is the set of all nonempty intersections of hyperplanes in $\mathcal{A}$, including V itself as the intersection over the empty set, ordered by reverse inclusion. Define the order relationship $x \leq y$ in $L(\mathcal{A})$ if $x \supseteq y$ in V.

Let G be a graph with vertex set $V = \{v_1, \ldots, v_n\}$ and edge set E. The *graphical arrangement* $\mathcal{A}_G$ in K^n is the arrangement with hyperplane $x_i - x_j = 0, v_i v_j \in E$. We will use poset notation and terminology from [7, Ch. 3]. In particular, the intersection poset of the graphical arrangement $\mathcal{A}_G$ (or of any central arrangement) is geometric. (An arrangement $\mathcal{A}$ is central if $\bigcap_{H \in \mathcal{A}} H \neq \emptyset$.) Let $2^{\mathbb{P}}$ denote the set of all subsets of $\mathbb{P}$, and let $\psi : V \to 2^{\mathbb{P}}$ satisfy $|\psi(v)| < \infty$ for all $v \in V$. Define the *ψ-graphical arrangement* $\mathcal{A}_{G,\psi}$ to be the arrangement in $\mathbb{R}^n$ with hyperplanes $x_i = x_j$ whenever $v_i v_j \in E$, together with $x_i = \alpha_j$ if $\alpha_j \in \psi(v_i)$.

In general, $\mathcal{A}_{G,\psi}$ is not a central arrangement and the intersection poset $L(\mathcal{A}_{G,\psi})$ of $\mathcal{A}_{G,\psi}$ is not a geometric lattice. Instead of $\mathcal{A}_{G,\psi}$ we consider the cone $c(\mathcal{A}_{G,\psi})$ with coordinates $x_1, \ldots, x_n, y$. The *cone ψ-graphical arrangement* $c(\mathcal{A}_{G,\psi})$ is the arrangement with hyperplanes $x_i = x_j$ whenever $v_i v_j \in E$, together with $y = 0$ and $x_i = \alpha_j y$ if $\alpha_j \in \psi(v_i)$.

An element x of a geometric lattice L is *modular* if $\mathrm{rk}(x) + \mathrm{rk}(y) = \mathrm{rk}(x \wedge y) + \mathrm{rk}(x \vee y)$ for all $y \in L$, where rk denotes the rank function of L. A geometric lattice L is *supersolvable* if there exists a *modular maximal chain*, i.e., a maximal chain $\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = \hat{1}$ such that each x_i is modular. A central arrangement $\mathcal{A}$ is *supersolvable* if its intersection lattice $L(\mathcal{A})$ is supersolvable.

A graph is *chordal* if each of its cycles of four or more vertices has a chord, which is an edge that is not part of the cycle but connects two vertices of the cycle. Equivalently, every induced cycle in the graph should have exactly three vertices. A graphical arrangement $\mathcal{A}_G$ is supersolvable if and only if G is a chordal graph [6, Cor. 4.10].

It is well known that the elements X_π of $L(\mathcal{A}_G)$ correspond to the connected partitions π of $V(G)$, i.e., the partitions $\pi = \{B_1, \ldots, B_k\}$ of $V(G)$ such that the restriction of G to each block B_i is connected.

We have $X_\pi \leq X_\sigma$ in $L(\mathcal{A}_G)$ if and only if every block of π is contained in a block of σ. Hence $L(\mathcal{A}_G)$ is isomorphic to an induced subposet L_G of Π_n, the lattice of partitions of the set $\{1, 2, \ldots, n\}$. From the definition of $L(c(\mathcal{A}_{G,\psi}))$ it is easy to see that $L(\mathcal{A}_G)$ is an interval of $L(c(\mathcal{A}_{G,\psi}))$, namely, the interval from the bottom element $\hat{0}$ (the ambient space K^n) to the intersection of all the hyperplanes $x_i = x_j$ of $c(\mathcal{A}_{G,\psi})$. For brevity, an element

$$X_\sigma = (x_1, \ldots, x_{i-1}, \alpha_i y, x_{i+1}, \ldots, x_{j-1}, \alpha_i y, x_{j+1}, \ldots, x_n, y)$$

($\alpha_i \in \psi(v_i)$ or $\alpha_i \in \psi(v_j)$) of $L(c(\mathcal{A}_{G,\psi}))$ is written as $\sigma : v_i = v_j = \alpha_i y$, or more briefly as $\sigma = \{v_i v_j \alpha_i y\}$, and an element

$$X_\delta = (x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_n, 0)$$

is written as $\delta : v_i = v_j = y = 0$, or more briefly as $\delta = \{v_i v_j y0\}$. The following sufficient condition for the supersolvability of a ψ-graphical arrangement is stated in [8] without proof.

Theorem 1 *Let (G, ψ) be as above. Suppose that we can order the vertices of G as $v_1, v_2, \ldots, v_n$ such that:*

(1) *v_{i+1} connects to previous vertices along a clique (so by Lemma 1 below G is chordal).*

(2) *If $i < j$ and v_i is adjacent to v_j, then $\psi(v_j) \subseteq \psi(v_i)$.*

Then $\mathcal{A}_{G,\psi}$ is supersolvable.

Proof To prove that $\mathcal{A}_{G,\psi}$ is supersolvable we need to find a modular maximal chain in $L(c(\mathcal{A}_{G,\psi}))$. We will show that a modular maximal chain is given by $\hat{0} < \pi_1 < \cdots < \pi_n < \hat{1}$, where $\pi_i = \{v_1 v_2 \cdots v_{i-1} y0\}$. First we prove that $\pi_n = \{v_1 v_2 \cdots v_{n-1} y0\}$ is a modular element. For any $\sigma = \{B_1, B_2, \ldots, B_t\} \in L(c(\mathcal{A}_{G,\psi}))$, we only need to consider the block B_i which contains v_n. If $B_i = \{v_n\}$ then $\sigma < \pi_n$. Hence $\mathrm{rk}(\pi_n) + \mathrm{rk}(\sigma) = \mathrm{rk}(\pi_n \wedge \sigma) + \mathrm{rk}(\pi_n \vee \sigma)$.

If $B_i = \{v_{i_1} \cdots v_{i_m} v_n\}$ then $\pi_n \vee \sigma = \hat{1}$. Since v_n connects to previous vertices along a clique, the block $B_i' = \{v_{i_1} \cdots v_{i_m}\}$ exists. Then $\pi_n \wedge \sigma = \{B_1, \ldots, B_{i-1}, B_{i+1}, \ldots, B_t, B_i', v_n\}$. Hence $\mathrm{rk}(\pi_n \wedge \sigma) = \mathrm{rk}(\sigma) - 1$ and $\mathrm{rk}(\pi_n) + \mathrm{rk}(\sigma) = \mathrm{rk}(\pi_n \wedge \sigma) + \mathrm{rk}(\pi_n \vee \sigma)$.

If $B_i = \{v_{i_1} \cdots v_{i_m} v_n y0\}$ then $\pi_n \vee \sigma = \hat{1}$ and

$$\pi_n \wedge \sigma = \{B_1, \ldots, B_{i-1}, B_{i+1}, \ldots, B_t, v_{i_1} \cdots v_{i_m} y0, v_n\}.$$

Hence $\mathrm{rk}(\pi_n \wedge \sigma) = \mathrm{rk}(\sigma) - 1$ and $\mathrm{rk}(\pi_n) + \mathrm{rk}(\sigma) = \mathrm{rk}(\pi_n \wedge \sigma) + \mathrm{rk}(\pi_n \vee \sigma)$.

If $B_i = \{v_{i_1} \cdots v_{i_m} v_n \alpha_j y\}$ ($\alpha_j \in \psi(v_{i_j}), 1 \le j \le m$, or $\alpha_j \in \psi(v_n)$) then $\pi_n \vee \sigma = \hat{1}$. Since $\psi(v_n) \subseteq \psi(v_{i_j})$ if $v_{i_j} v_n \in E$, i.e., if $\alpha_j \in \psi(v_n)$, we have $\alpha_j \in \psi(v_{i_j})$. Hence the block $B_i' = \{v_{i_1} \cdots v_{i_m} \alpha_j y\}$ exists and $\pi_n \wedge \sigma = \{B_1, \ldots, B_{i-1}, B_{i+1}, \ldots, B_t, B_i', v_n\}$. Then $\mathrm{rk}(\pi_n \wedge \sigma) = \mathrm{rk}(\sigma) - 1$ and $\mathrm{rk}(\pi_n) + \mathrm{rk}(\sigma) = \mathrm{rk}(\pi_n \wedge \sigma) + \mathrm{rk}(\pi_n \vee \sigma)$.

Hence we get that $\pi_n = \{v_1 v_2 \cdots v_{n-1} y0\}$ is a modular element. Now if $\pi_{n-1} = \{v_1 v_2 \cdots v_{n-2} y0\}$ is modular in the interval $[\hat{0}, \pi_n]$, then it is modular in $L(c(\mathcal{A}_{G,\psi}))$ [6, Prop. 4.10(b)]. Therefore we just need to show that π_{n-1} is modular in the interval $[\hat{0}, \pi_n]$.

Since all elements σ in $[\hat{0}, \pi_n]$ must satisfy that σ has a block $B_i = \{v_n\}$, we can ignore the block $B_i = \{v_n\}$. In the same way we can get that $\pi_{n-1} = \{v_1 v_2 \cdots v_{n-2} y0\}$ is a modular element in the interval $[\hat{0}, \pi_n]$. Continuing the procedure, we get the modular maximal chain $\hat{0} < \pi_1 < \cdots < \pi_n < \hat{1}$.

Our main result is the converse to Theorem 1.

Theorem 2 *The sufficient condition in Theorem 1 for the supersolvability of $\mathcal{A}_{G,\psi}$ is also necessary.*

Before we prove Theorem 2, the following two results of Dirac [1] are required. A vertex is *simplicial* in a graph if its neighbors form a complete subgraph. A graph is *recursively simplicial* if it consists of a single vertex, or if it contains a simplicial vertex v and when v is removed the subgraph that remains is recursively simplicial. It is well-known and easy to see that if G is recursively simplicial and v is *any* vertex, then $G - v$ is recursively simplicial

Lemma 1 *G is chordal if and only if G is recursively simplicial.*

Lemma 2 *Every chordal graph G that is not a complete graph has at least two non-adjacent simplicial vertices.*

Proof (of Theorem 2) Condition (1) is easy to check, because $L(\mathcal{A}_G)$ is an interval of $L(c(\mathcal{A}_{G,\psi}))$. Since intervals of supersolvable lattices are supersolvable ([5, Prop. 3.2]), we have that $L(c(\mathcal{A}_G))$ is supersolvable. Hence by [5, Prop. 2.8] G is chordal.

By Lemma 2 we know that there are at least two nonadjacent simplicial vertices in the chordal graph G. Suppose that there is a simplicial vertex, say v_{i_n}, which satisfies the following condition:

$$\psi(v_{i_n}) \subseteq \psi(v_{i_j}) \text{ for all } v_{i_j}v_{i_n} \in E. \tag{1.1}$$

Then we label v_{i_n} as v_n and remove this vertex. By Lemma 1 we know that the remaining graph is still recursively simplicial. Continuing in this way, suppose that there is a simplicial vertex, which we label as v_{n-1} and then remove it. Continue this procedure. If condition (2) is not necessary then that means there exists one step m in the above procedure such that all remaining simplicial vertices do not satisfy the condition (1.1). Then we will show that there is no modular maximal chain in $L(c(\mathcal{A}_{G,\psi}))$.

Next, we show that among all the coatoms only $\sigma_i = \{v_{i_1}v_{i_2}\cdots v_{i_{n-1}}y0\}$ and $\delta_i = \{v_1v_2\cdots v_n\alpha_i y\}, \alpha_i \in \psi(v_i), 1 \le i \le n$ could be modular elements of $L(c(\mathcal{A}_{G,\psi}))$. We claim that a coatom is not modular if it has more than two blocks or it has two blocks but the cardinalities of both of the blocks are greater than 1.

First, it is easy to check that any coatom σ is not modular if it has more than two blocks. Suppose $\sigma = \{A, B, C\}$ is a coatom. Since $\mathrm{rk}(\sigma) = n - 1$, i.e., $\dim(\sigma) = 1$, A, B and C can only be $\{v_{i_1}v_{i_2}\cdots v_{i_{j_i}}\alpha_i y\}$ where $i = 1, 2, 3$ and $\alpha_i \in \psi(v_{i_m}), m = 1, 2, \ldots, j_i$. Then $\gamma = \{v_1v_2\cdots v_n, y0\}$, $\mathrm{rk}(\sigma) = \mathrm{rk}(\gamma) = n - 1$, $\mathrm{rk}(\sigma \vee \gamma) = n$ but $\mathrm{rk}(\sigma \wedge \gamma) < n - 2$. Hence σ is not modular.

Moreover if $\sigma = \{A, B\}$ is a coatom such that $|A| > 1$ and $|B| > 1$ then σ is also not modular. Without loss of generality assume that there exist $u, v \in A$ and $u', v' \in B$ such that $u \neq v'$ and $u' \neq v$, $uu' \in E(G)$, and $vv' \in E(G)$. Let $\gamma = \{(A \cup u')\backslash v, (B \cup v)\backslash u'\}$. Then $\mathrm{rk}(\sigma) = \mathrm{rk}(\gamma) = n - 1, \mathrm{rk}(\sigma \vee \gamma) = n$ but $\mathrm{rk}(\sigma \wedge \gamma) < n - 2$. Hence σ is not modular.

Therefore, among all coatoms, only $\sigma_i = \{v_{i_1} v_{i_2} \cdots v_{i_{n-1}} y0\}$ and

$$\delta_i = \{v_1 v_2 \cdots v_n \alpha_i y, \alpha_i \in \psi(v_i), 1 \leq i \leq n\}$$

could be modular elements. Similarly, among all the elements which σ_i covers, only $\{(v_{i_1} v_{i_2} \cdots v_{i_{n-1}} \backslash v_{i_j}) y0\}$ could be modular.

If v_{i_n} is not a simplicial vertex then we show that σ_i is not modular. Without loss of generality assume $v_{i_s} v_{i_n} \in E$ and $v_{i_t} v_{i_n} \in E$ but $v_{i_s} v_{i_t} \notin E$. Let $\gamma = \{(v_{i_1} v_{i_2} \ldots v_{i_{n-1}} \backslash v_{i_s} v_{i_t}) y0, v_{i_s} v_{i_t} v_{i_n}\}$. Then $\mathrm{rk}(\sigma) = \mathrm{rk}(\gamma) = n-1, \mathrm{rk}(\sigma \vee \gamma) = n$ but $\mathrm{rk}(\sigma \wedge \gamma) < n-2$. Hence σ_i is not modular if v_{i_n} is not a simplicial vertex.

We now show that if v_{i_n} is a simplicial vertex but does not satisfy condition (1.1), then σ_i is not modular. Without loss of generality assume that $\alpha_i \in \psi(v_{i_n})$ but $\alpha_i \notin \psi(v_{i_j})$ for $v_{i_j} v_{i_n} \in E$. Then $\gamma = \{v_{i_j} v_{i_n} \alpha_i y\}$, $\mathrm{rk}(\sigma_i) = n-1, \mathrm{rk}(\gamma) = 2, \mathrm{rk}(\sigma_i \vee \gamma) = n$ but $\mathrm{rk}(\sigma_i \wedge \gamma) = 0$. From the above discussion, if condition (2) is not necessary then there exists one step m such that all remaining simplicial vertices do not satisfy condition (1.1). That means that all $\{v_{i_1} v_{i_2} \cdots v_{i_{n-m}} y0\}$ are not modular elements. Hence there is no modular maximal chain from $\hat{0}$ to σ_i.

We now show that if δ_i is modular then $\alpha_i \in \psi(v_i)$ for all $i \in [n]$. Equivalently, we show that if there exists some v_m such that $\alpha_i \notin \psi(v_m)$, then δ_i is not modular. Let $\gamma = \{v_1 v_2 \cdots v_n \backslash v_m, v_m y0\}$. Hence $\mathrm{rk}(\delta_i) = \mathrm{rk}(\gamma) = n-1, \mathrm{rk}(\delta_i \vee \gamma) = n$ but $\mathrm{rk}(\delta_i \wedge \gamma) < n-2$. From the above discussion, if condition (2) is not necessary then there are at least two nonadjacent simplicial vertices, say v_s and v_t, which do not satisfy condition (1.1). It means that there exist $\alpha_s \in \psi(v_s), \alpha_t \in \psi(v_t)$ and $\alpha_s, \alpha_t \neq \alpha_i$. If $\alpha_s = \alpha_t$ then let $\gamma = \{(v_1 v_2 \cdots v_n \backslash v_s v_t) \alpha_i y, v_s v_t \alpha_s y\}$. Hence $\mathrm{rk}(\delta_i) = \mathrm{rk}(\gamma) = n-1, \mathrm{rk}(\delta_i \vee \gamma) = n$ but $\mathrm{rk}(\delta \wedge \gamma) < n-2$, so δ_i is not modular.

If $\alpha_s \neq \alpha_t$ then let $\gamma = \{v_t \alpha_t y, v_s \alpha_s y, (v_1 v_2 \cdots v_n \backslash v_t v_s) \alpha_i y\}$. Now $\mathrm{rk}(\delta_i) = \mathrm{rk}(\gamma) = n-1, \mathrm{rk}(\delta_i \vee \gamma) = n$ but $\mathrm{rk}(\delta_i \wedge \gamma) < n-2$, so δ_i is not modular.

Therefore if there does not exist a labeling such that conditions (1) and (2) holds, then we can't find a modular maximal chain in $L(c(\mathcal{A}_G))$. Hence the proof is complete.

We call $v_1, \ldots, v_n$ a *vertex elimination order* for G if v_{i+1} connects to previous vertices along a clique. For any supersolvable arrangement $\mathcal{A}$ of rank n the characteristic polynomial of $\mathcal{A}$ (defined, e.g., in [6, §1.3] or [7, §3.11.2]) factors as $\chi_G(q) = \prod_{i=1}^n (q - a_i)$, where $a_1, \ldots, a_n$ are nonnegative integers, called the *exponents* of $\mathcal{A}$. There is a simple combinatorical interpretation of the exponents of $\mathcal{A}(G)$ when G is chordal.

Proposition 1 *[2, Lemma 3.4] Let G be a chordal graph with vertex elimination order $\{v_1, \ldots, v_n\}$. For $1 \leqslant i \leqslant n$ let b_i be the degree of v_i in the graph $G - \{v_n, \ldots, v_{i+1}\}$. Then $\{b_1, \ldots, b_n\}$ are the exponents of the supersolvable arrangement $\mathcal{A}(G)$.*

It is not hard to get a similar property for the supersolvable arrangement $\mathcal{A}_{G,\psi}$. We omit the proof of this proposition.

Proposition 2 *Let (G, ψ) be a chordal graph with vertex elimination order $\{v_1, \ldots, v_n\}$. Assume that for any $v_iv_j \in E(G)$ such that if $i < j$, we have $\psi(v_j) \subseteq \psi(v_i)$. For $1 \leqslant i \leqslant n$ let b_i be the sum of $|\psi(v_i)|$ and the degree of v_i in the graph $G - \{v_n, \ldots, v_{i+1}\}$. Then $\{b_1, \ldots, b_n\}$ are the exponents of the supersolavable arrangement $\mathcal{A}_{G,\psi}$.*

There is another conjecture in [8]. It is well known that every supersolvable arrangement is free (in the sense of Terao [3, §6.3]) and every free graphical arrangement is supersolvable. Thus the second author proposed the following conjecture.

Conjecture 1 If $\mathcal{A}_{G,\psi}$ is a free ψ-graphical arrangement, then $\mathcal{A}_{G,\psi}$ is supersolvable.

We are unable to prove this conjecture, but we do have the following weaker result, which we simply state without proof. The proof involves the inheritance of freeness under localization of arrangements and a result of Yoshinaga [9] on the freeness of 3-arrangements.

Theorem 3 *The ψ-graphical arrangement $\mathcal{A}_{G,\psi}$ is not free if there is an edge $v_iv_j \in E(G)$ such that $\psi(v_i) \nsubseteq \psi(v_j)$ and $\psi(v_j) \nsubseteq \psi(v_i)$.*

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